# Spectral factorization in the non–stationary Wiener algebra

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**Abstract**

We define the non–stationary analogue of the Wiener algebra and prove a spectral factorization theorem in this algebra.

## 1 Introduction

In this paper we prove a spectral factorization theorem in the non–stationary analogue of the Wiener algebra. To set the problem into perspective and to present our result we first briefly review the case of operator–valued functions on the unit circle. Let $\mathcal{B}$ be a Banach algebra with norm $\|\cdot\|_{\mathcal{B}}$ and let $\mathcal{W}(\mathcal{B})$ denote the Banach algebra of functions of the form

$$f(e^{it}) = \sum_{n\in\mathbb{Z}} e^{int} f_n$$

where the $b_n$ are in $\mathcal{B}$ and such that $\|f\|_{\mathcal{W}(\mathcal{B})} \stackrel{\text{def.}}{=} \sum_{\mathbb{Z}} \|f_n\|_{\mathcal{B}} < \infty$. We set

$$\mathcal{W}_+(\mathcal{B}) = \{f \in \mathcal{W}(\mathcal{B}) \mid f_n = 0,\ n < 0\} \quad \text{and} \quad \mathcal{W}_-(\mathcal{B}) = \{f \in \mathcal{W}(\mathcal{B}) \mid f_n = 0,\ n > 0\}.$$

Inversion theorems in $\mathcal{W}(\mathcal{B})$ originate with the work of Wiener for the case $\mathcal{B} = \mathbb{C}$, and with the work of Bochner and Phillips in the general case; see [21] and [6] respectively. Gohberg and Leiterer studied in [17], [18] factorizations in $\mathcal{W}(\mathcal{B})$. A particular case of their results is:

**Theorem 1.1** *Let $W \in \mathcal{W}(\mathcal{B})$ and assume that $W(e^{it}) > 0$ for every real $t$. Then there exists $W_+ \in \mathcal{W}_+(\mathcal{B})$ such that $W_+^{-1} \in \mathcal{W}_+(\mathcal{B})$ and $W = W_+^* W_+$.*

See Step 3 in the proof of Proposition 3.1.

We recall that $W(e^{it}) \in \mathcal{B}$ for every real $t$ and that $W(e^{it}) > 0$ is understood in $\mathcal{B}$ (that is, $W(e^{it}) = X(t)^* X(t)$ for some $X(t) \in \mathcal{B}$ which is invertible in $\mathcal{B}$). For instance, when $\mathcal{B}$ is the space of bounded operators from a Hilbert space into itself, $W(e^{it}) > 0$ means that $W(e^{it})$ is a positive boundedly invertible operator.

We note that other settings, where $e^{it}$ is replaced by a strictly contractive $Z \in \mathcal{B}$, are also known; see in particular [19],[15] and [14].

In the present paper we consider the case when $\mathcal{B}$ is the space of block–diagonal operators from $\ell^2_{\mathcal{M}}$ itself (here $\mathcal{M}$ is a pre–assigned Hilbert space and $\ell^2_{\mathcal{M}}$ denotes the Hilbert space of square summable sequences with components in $\mathcal{M}$ and indexed by $\mathbb{Z}$) and $e^{it}$ is replaced by $Z$, the natural bilateral backward shift from $\ell^2_{\mathcal{M}}$ into itself. This setting is related to the theory of non-stationary linear systems (see [11]).

**Definition 1.2** *The non–stationary Wiener algebra $\mathcal{W}_{NS}$ consists of the set of operators in $\mathbf{L}(\ell^2_{\mathcal{M}})$ of the form $F = \sum_{\mathbb{Z}} Z^n F_{[n]}$ where the $F_{[n]}$ are diagonal operators such that*

$$\|F\|_{\mathcal{W}_{NS}} \stackrel{\text{def.}}{=} \sum_{\mathbb{Z}} \|F_{[n]}\| < \infty \tag{1.1}$$

*The element $F$ belongs to $\mathcal{W}^+_{NS}$ (resp. $\mathcal{W}^-_{NS}$) if $F_{[n]} = 0$ for $n < 0$ (resp. for $n > 0$).*

The *a priori* formal sum $F = \sum_{\mathbb{Z}} Z^n F_{[n]}$ actually converges in the operator norm because of (1.1). The fact that $\mathcal{W}_{NS}$ is a Banach algebra follows from the fact that $ZDZ^*$ and $Z^*DZ$ are diagonal operators when $D$ is a diagonal operator.

The main result of this paper is:

**Theorem 1.3** *Let $W \in \mathcal{W}_{NS}$ and assume that $W$ positive definite (as an operator from $\ell^2_{\mathcal{M}}$ into itself). Then there exists $W_+ \in \mathcal{W}^+_{NS}$ such that $W_+^{-1} \in \mathcal{W}^+_{NS}$ and $W = W_+^* W_+$.*

We note the following: as remarked in [11, p. 369], Arveson's factorization theorem (see [5]) implies that a positive definite $W$ can be factorized as $W = U^*U$, where $U$ and its inverse are upper triangular operators. The new point here is that if $W$ is in the non–stationary Wiener algebra, then so are $U$ and its inverse.

A special case of Theorem 1.3 when $W$ admits a realization is given in [11, Theorem 13.5 p. 369] with explicit formula for the spectral factor.

We now turn to the outline of the paper. It consists of four sections besides the introduction. In Section 2 we review the non–stationary (also called time–varying) setting. In particular we review facts on the Zadeh transform associated to a bounded upper triangular operator. In Section 3 we obtain a spectral factorization for the function $\sum_{\mathbb{Z}} e^{int} Z^n W_{[n]}$. In Section 4 we obtain a lower–upper factorization of that same function. Comparing the two factorizations lead to the proof of Theorem 1.3. This is done in the last section.

# 2 The non–stationary setting and the Zadeh transform

In this section we review the non–stationary setting. We follow the analysis and notations of [4] and [10]. Let $\mathcal{M}$ be a separable Hilbert space, "the coefficient space". As in [10, Section 1], the set of bounded linear operators from the space $\ell^2_{\mathcal{M}}$ of square summable sequences

with components in $\mathcal{M}$ into itself is denoted by $\mathcal{X}(\ell^2_{\mathcal{M}})$, or $\mathcal{X}$. The space $\ell^2_{\mathcal{M}}$ is taken with the standard inner product. Let $Z$ be the bilateral backward shift operator

$$(Zf)_i = f_{i+1}, \qquad i = \dots, -1, 0, 1, \dots$$

where $f = (\dots, f_{-1}, \boxed{f_0}, f_1, \dots) \in \ell^2_{\mathcal{M}}$. The operator $Z$ is unitary on $\ell^2_{\mathcal{M}}$ i.e. $ZZ^* = Z^*Z = I$, and

$$\pi^* Z^j \pi = \begin{cases} I_{\mathcal{M}} & \text{if} \quad j = 0 \\ 0_{\mathcal{M}} & \text{if} \quad j \neq 0. \end{cases}$$

where $\pi$ denote the injection map

$$\pi : u \in \mathcal{M} \to f \in \ell^2_{\mathcal{M}} \quad \text{where} \quad \begin{cases} f_0 = u \\ f_i = 0, \quad i \neq 0 \end{cases} .$$

We define the space of upper triangular operators by

$$\mathcal{U}\left(\ell^2_{\mathcal{M}}\right) = \left\{ A \in \mathcal{X}\left(\ell^2_{\mathcal{M}}\right) \,\middle|\, \pi^* Z^i A Z^{*j} \pi = 0 \quad \text{for} \quad i > j \right\},$$

and the space of lower triangular operators by

$$\mathcal{L}\left(\ell^2_{\mathcal{M}}\right) = \left\{ A \in \mathcal{X}\left(\ell^2_{\mathcal{M}}\right) \,\middle|\, \pi^* Z^i A Z^{*j} \pi = 0 \quad \text{for} \quad i < j \right\}.$$

The space of diagonal operators $\mathcal{D}(\ell^2_{\mathcal{M}})$ consists of the operators which are both upper and lower triangular. As for the space $\mathcal{X}$, we usually denote these spaces by $\mathcal{U}$, $\mathcal{L}$ and $\mathcal{D}$.

Let $A^{(j)} = Z^{*j} A Z^j$ for $A \in \mathcal{X}$ and $j = \dots, -1, 0, 1, \dots$; note that $\left(A^{(j)}\right)_{st} = A_{s-j,t-j}$ and that the maps $A \mapsto A^{(j)}$ take the spaces $\mathcal{L}$, $\mathcal{D}$, $\mathcal{U}$ into themselves. Clearly, for $A$ and $B$ in $\mathcal{X}$ we have that $(AB)^{(j)} = A^{(j)} B^{(j)}$ and $A^{(j+k)} = \left(A^{(j)}\right)^{(k)}$.

In [4] it is shown that for every $F \in \mathcal{U}$, there exists a unique sequence of operators $F_{[j]} \in \mathcal{D}$, $j = 0, 1, \dots$ such that

$$F - \sum_{j=0}^{n-1} Z^j F_{[j]} \in Z^n_{\mathcal{M}} \mathcal{U}.$$

In fact, $\left(F_{[j]}\right)_{ii} = F_{i-j,i}$ and we can formally represent $F \in \mathcal{U}$ as the sum of its diagonals

$$F = \sum_{n=0}^{\infty} Z^n F_{[n]}.$$

More generally one can associate to an element $F \in \mathcal{X}$ a sequence of diagonal operators such that, formally $F = \sum_{\mathbb{Z}} Z^n F_{[n]}$. Recall the well known fact that even when $F$ is a bounded operator the formal sums $\sum_{n=0}^{\infty} Z^n F_{[n]}$ and $\sum_{-\infty}^{0} Z^n F_{[n]}$ need not define bounded operators. See e.g. [11, p. 29] for a counterexample.

When the operator $F$ is in the Hilbert–Schmidt class (we will use the notation $F \in \mathcal{X}_2$) the above representation is not formal but converges both in operator and Hilbert–Schmidt

norm. Indeed, each of the diagonal operator $F_{[n]}$ is itself a Hilbert–Schmidt operator and we have:

$$\|F\|_{\mathcal{X}_2}^2 = \sum_{n=0}^{\infty} \|F_{[n]}\|_{\mathcal{X}_2}^2 < \infty \tag{2.1}$$

and

$$\begin{aligned}
\|F - \sum_{-M}^{N} Z^n F_{[n]}\|^2 &\leq \|F - \sum_{-M}^{N} Z^n F_{[n]}\|_{\mathcal{X}_2}^2 \\
&= \sum_{-\infty}^{-M-1} \|F_{[n]}\|_{\mathcal{X}_2}^2 + \sum_{N+1}^{\infty} \|F_{[n]}\|_{\mathcal{X}_2}^2 \\
&\to 0 \quad \text{as} \quad N, M \to \infty.
\end{aligned}$$

Here we used the fact that the operator norm is less that the Hilbert–Schmidt norm:

$$\|F\| \leq \|F\|_{\mathcal{X}_2}. \tag{2.2}$$

See e.g. [8, EVT V.52].

**Definition 2.1** *The Hilbert space of upper triangular (resp. diagonal) Hilbert–Schmidt operators will be denoted by* $\mathcal{U}_2$ *(resp. by* $\mathcal{D}_2$*).*

As already mentioned, the non–stationary Wiener algebra is another example where the formal power series converges in the operator norm.

**Proposition 2.2** *The space* $\mathcal{W}_{NS}$ *endowed with* $\|\cdot\|_{NS}$ *is a Banach algebra.*

**Proof:** Let $F$ and $G$ be in $\mathcal{W}_{NS}$ with representations

$$F = \sum_{\mathbb{Z}} Z^n F_{[n]} \quad \text{and} \quad G = \sum_{\mathbb{Z}} Z^n G_{[n]}.$$

Then the family $Z^m F_{[n]}^{(m-n)} G_{[m-n]}$ is absolutely convergent since $\|D\| = \|D^{(j)}\|$ for every diagonal operator $D$ and integer $j \in \mathbb{Z}$. It is therefore commutatively convergent (see [7, Corollaire 1 p. TG IX.37]) and we can write

$$\begin{aligned}
FG &= \left(\sum_{\mathcal{Z}} Z^n F_{[n]}\right)\left(\sum_{\mathcal{Z}} Z^p G_{[p]}\right) \\
&= \sum_{n,p\in\mathbb{Z}} Z^n F_{[n]} Z^p G_{[p]} \\
&= \sum_{n,p\in\mathbb{Z}} Z^{n+p} F_{[n]}^{(p)} G_{[p]} \\
&= \sum_{m\in\mathbb{Z}} Z^m \left(\sum_{n\in\mathbb{Z}} F_{[n]}^{(m-n)} G_{[m-n]}\right) \\
&= \sum_{m\in\mathbb{Z}} Z^m (FG)_{[m]}
\end{aligned}$$

with

$$(FG)_{[m]} = \sum_{n\in\mathbb{Z}} F_{[n]}^{(m-n)} G_{[m-n]}.$$

This exhibits $FG$ as an element of $\mathcal{W}_{NS}$. The Banach algebra norm inequality holds in $\mathcal{W}_{NS}$ since

$$\sum_{m\in\mathbb{Z}} \|(FG)_{[m]}\| \le \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \|F_{[n]}\| \cdot \|G_{[m-n]}\| = \|F\|_{\mathcal{W}_{NS}} \|G\|_{\mathcal{W}_{NS}}.$$

$\square$

**Definition 2.3** *Let $U \in \mathcal{U}$ with formal representation $U = \sum_{n=0}^{\infty} Z^n U_{[n]}$. The Zadeh transform of $U$ is the $\mathcal{X}$–valued function defined by*

$$U(z) = \sum_{n=0}^{\infty} z^n Z^n U_{[n]}, \qquad z \in \mathbb{D}. \tag{2.3}$$

We note that the series (2.3) converges in the operator norm for every $z \in \mathbb{D}$ and that $F(z)$ is called in [12] the symbol of $F$; see [12, p. 135].

**Theorem 2.4** *Let $U$, $U_1$ and $U_2$ be upper-triangular operators. Then,*

$$\|U(z)\| \le \|U\|, \qquad z \in \mathbb{D} \tag{2.4}$$

*and*

$$(U_1U_2)(z) = U_1(z)U_2(z). \tag{2.5}$$

**Proof:** A proof of the first claim can be found in [12, Theorem 5.5 p 136]. The key ingredient in the proof is that every upper triangular contraction is the characteristic function of a unitary colligation; see [12, Theorem 5.3 p. 135]. To prove the second claim we remark, as in [12, p. 136] that

$$U(z) = \Lambda(z)U\Lambda(z)^{-1} \tag{2.6}$$

where $z \neq 0$ and where $\Lambda(z)$ denotes the *unbounded* diagonal operator defined by

$$\Lambda(z) = \mathrm{diag}\ \begin{pmatrix} \cdots & z^2 I_{\mathcal{M}} & zI_{\mathcal{M}} & I_{\mathcal{M}} & z^{-1}I_{\mathcal{M}} & z^{-2}I_{\mathcal{M}} & \cdots \end{pmatrix}. \tag{2.7}$$

Of course some care is needed with (2.6). What is really meant is that the *a priori* unbounded operator on the right coincides with the bounded operator on the left on a dense set (for instance on the set of sequences with finite support):

$$\Lambda(z)U\Lambda(z)^{-1}u = U(z)u \tag{2.8}$$

where $u \in \ell^2_{\mathcal{M}}$ is a sequence with finite support.

We now proceed as follows to prove (2.5). We start with a sequence $u$ as above. Then:

$$\begin{aligned} \Lambda(z)^{-1}u \in \ell^2_{\mathcal{M}} & \quad \text{(since } u \text{ has finite support)} \\ U_2\Lambda(z)^{-1}u \in \ell^2_{\mathcal{M}} & \quad \text{(since dom } U_2 = \ell^2_{\mathcal{M}}) \\ \Lambda(z)U_2\Lambda(z)^{-1}u \in \ell^2_{\mathcal{M}} & \quad \text{(by (2.8))} \\ \Lambda(z)^{-1}\Lambda(z)U_2\Lambda(z)^{-1}u \in \ell^2_{\mathcal{M}} & \quad \text{since it is equal to } U_2\Lambda(z)^{-1}u \end{aligned}$$

and so (still for sequences with finite support)

$$U_1\Lambda(z)^{-1}\Lambda(z)U_2\Lambda(z)^{-1}u = U_1U_2\Lambda(z)^{-1}u$$

and applying (2.8) we conclude that

$$\Lambda(z)U_1\Lambda(z)^{-1}\Lambda(z)U_2\Lambda(z)^{-1}u = \Lambda(z)F_1F_2\Lambda(z)^{-1}u = (F_1F_2)(z)u \in \ell^2_{\mathcal{M}}.$$

These same equalities prove (2.5). □

For a discussion and references on the Zadeh transform we refer to [1, p. 255–257]. The Zadeh–transform was used in [2], [3] to attack some problems where unbounded operators appear. Here our point of view is a bit different.

We now extend the Zadeh transform to operators in $\mathcal{W}_{NS}$ and define

$$W(e^{it}) = \sum_{\mathbb{Z}} e^{int} Z^n W_{[n]}$$

for $W \in \mathcal{W}_{NS}$ with representation $W = \sum_{\mathbb{Z}} Z^n W_{[n]}$. We obtain a function which is continuous on the unit circle and it is readily seen that

$$\|W(e^{it})\| \le \|W\|_{\mathcal{W}_{NS}} \quad \text{and} \quad (W_1W_2)(e^{it}) = W_1(e^{it})W_2(e^{it})$$

for $W, W_1$ and $W_2$ in $\mathcal{W}_{NS}$. To prove the second equality it suffices to note that $\Lambda(e^{it})$ is now a unitary operator and that

$$W(e^{it}) = \Lambda(e^{it})W\Lambda(e^{it})^{-1}. \tag{2.9}$$

# 3 Spectral factorization of $W(e^{it})$

In this section we develop the first step in the proof of Theorem 1.3.

**Proposition 3.1** *Let $W \in \mathcal{W}_{NS}$ and assume that $W > 0$. Then $W(e^{it}) > 0$ for every real $t$ and there exists a $\mathcal{X}$–valued function*

$$X(z) = \sum_{n=0}^{\infty} z^n X_n$$

*with the following properties:*

1. *The $X_n \in \mathcal{X}$ and $\sum_{n=0}^{\infty} \|X_n\| < \infty$ (that is, $X \in \mathcal{W}_+(\mathcal{X})$).*
2. *$X$ is invertible and its inverse belongs to $\mathcal{W}_+(\mathcal{X})$.*
3. *$W(e^{it}) = X(e^{it})^* X(e^{it})$.*

**Proof:** We write $W = \text{Re}\ \Phi$ where $\Phi = W_{[0]} + 2\sum_{n=1}^{\infty} Z^n W_{[n]}$ and proceed in a number of steps (note that $\Phi$ is a bounded upper triangular operator since $\sum_{n\geq 0} \|W_{[n]}\| < \infty$).

STEP 1. *The operator* $(I+\Phi)$ *is invertible in* $\mathcal{U}$ *and*

$$(I+\Phi)^{-1}(z) = (I+\Phi(z))^{-1}. \tag{3.1}$$

Consider the multiplication operator $M_\Phi : \mathcal{U}_2 \longrightarrow \mathcal{U}_2$, defined by $M_\Phi F = \Phi F$. Then, for every $F, G \in \mathcal{U}_2$, we have

$$\langle \Phi F, G\rangle_{\mathcal{U}_2} = \langle \Phi F, G\rangle_{\mathcal{X}_2} = \langle F, \Phi^* G\rangle_{\mathcal{X}_2}.$$

Note that $I + M_\Phi$ is a multiplication operator, as well: $I + M_\Phi = M_\Psi$, where $\Psi = \Phi + I$. Hence, for every $F \in \mathcal{U}_2$, we have:

$$\begin{aligned}
\langle M_\Psi F, M_\Psi F\rangle_{\mathcal{U}_2} &= \langle \Phi F, \Phi F\rangle_{\mathcal{U}_2} + \langle \Phi F, F\rangle_{\mathcal{U}_2} + \langle F, \Phi F\rangle_{\mathcal{U}_2} + \langle F, F\rangle_{\mathcal{U}_2} \\
&= \langle \Phi F, \Phi F\rangle_{\mathcal{U}_2} + \langle (\Phi + \Phi^*) F, F\rangle_{\mathcal{X}_2} + \langle F, F\rangle_{\mathcal{U}_2} \\
&\geq \langle F, F\rangle_{\mathcal{U}_2}.
\end{aligned}$$

In particular, $M_\Psi$ is one-to-one.

In the same manner,

$$\begin{aligned}
\langle M_\Psi^* F, M_\Psi^* F\rangle_{\mathcal{U}_2} &= \langle M_\Phi^* F, M_\Phi^* F\rangle_{\mathcal{U}_2} + \langle M_\Phi^* F, F\rangle_{\mathcal{U}_2} + \langle F, M_\Phi^* F\rangle_{\mathcal{U}_2} + \langle F, F\rangle_{\mathcal{U}_2} \\
&\geq \langle \Phi F, F\rangle_{\mathcal{U}_2} + \langle F, \Phi F\rangle_{\mathcal{U}_2} + \langle F, F\rangle_{\mathcal{U}_2} \\
&\geq \langle F, F\rangle_{\mathcal{U}_2},
\end{aligned}$$

and, in particular, $M_\Psi$ is onto (see [9, p. 30] if need be). Therefore, by the open mapping theorem $M_\Psi$ is invertible.

Analogous reasoning shows that $\Psi : \ell^2_{\mathcal{M}} \longrightarrow \ell^2_{\mathcal{M}}$ is invertible, as well. Moreover, the multiplication operator $M_{\Psi^{-1}} : \mathcal{X}_2 \longrightarrow \mathcal{X}_2$ preserves the subspace $\mathcal{U}_2$:

$$M_{\Psi^{-1}|\mathcal{U}_2} = M_{\Psi^{-1}|\mathcal{U}_2} M_\Psi M_\Psi^{-1} = M_\Psi^{-1}.$$

Thus $\Psi^{-1} \in \mathcal{U}$. Since $(I+\Phi)^{-1} \in \mathcal{U}$ and using (2.5) we have:

$$\left((I+\Phi)((I+\Phi)^{-1}\right)(z) = (I+\Phi(z))(I+\Phi)^{-1}(z)$$

and hence we obtain (3.1). $\square$

STEP 2. *It holds that* Re $\Phi(z) > 0$ *for* $z \in \mathbb{D}$.

Indeed, the operator $S = (I+\Phi)^{-1}(I-\Phi)$ is upper triangular and $\|S\| < 1$. By (2.4), $\|S(z)\| < 1$ for all $z \in \mathbb{D}$ and thus Re $(S(z) - I)(S(z)+I)^{-1} > 0$.

STEP 3. *To conclude it suffices to apply the results of [16] to the function $W(e^{it})$.*

Indeed, consider the Toepliz operator with symbol $W(e^{it})$. It is self–adjoint and invertible since $W(e^{it}) > 0$. Thus by [16, Theorem 0.4 p. 106], $W(e^{it}) = W_-(e^{it})W_+(e^{it})$ where $W_+$ and its inverse are in $\mathcal{W}_+(\mathcal{X})$ and $W_-$ and its inverse are in $\mathcal{W}_-(\mathcal{X})$. By uniqueness of the factorization $W_+(e^{it}) = MW_-(e^{it})^*$. The operator $M$ is strictly positive and one deduces the factorization result by replacing $W_+(e^{it})$ by $W_+(e^{it})M^{1/2}$. □

# 4 Lower-upper factorization of $W(e^{it})$

We now use Arveson factorization theorem to obtain another factorization of $W(e^{it})$.

**Proposition 4.1** *Let $W \in \mathcal{X}$ be strictly positive. Then there exists $U \in \mathcal{U}$ such that $W = U^*U$*

**Proof:** It suffices to apply the result of [5] (see also [13, p. 88]) to the nest algebra defined by the resolution of the identity

$$E_n(\ldots, f_{-1}, \boxed{f_0}, f_1, \ldots) = (\ldots, f_{n-1}, f_n)$$

□

**Proposition 4.2** *Let $U, V \in \mathcal{U}$ with formal expansions $U = \sum_{n=0}^{\infty} Z^n U_{[n]}$ and $V = \sum_{n=0}^{\infty} Z^n V_{[n]}$. Let $z = re^{it} \in \mathbb{D}$. Then*

$$\Omega(z) = V(z)^*U(z) = \sum_{\mathbb{Z}} e^{imt} Z^m \Omega_{[m]}(r)$$

*where*

$$\Omega_{[m]}(r) = \begin{cases} \sum_{p=0}^{\infty} r^{2p} V_{[p]}^* U_{[p]} & \text{if} \quad m = 0 \\ \sum_{p=m}^{\infty} r^{2n-m} (V_{[p-m]}^*)^{(m)} U_{[p]} & \text{if} \quad m > 0 \end{cases}$$

*and $\Omega_{[-m]} = (\Omega_{[m]}^*)^{(-m)}$ and the sums converge in the operator norm.*

**Proof:** Since the series $U(z) = \sum_{n=0}^{\infty} r^n e^{int} Z^n U_{[n]}$ and $V(z) = \sum_{n=0}^{\infty} r^n e^{int} Z^n V_{[n]}$ converge in the operator norm this is an easy computation which is omitted.

□

The case $r = 1$ is more involved.

**Proposition 4.3** *Let $U, V \in \mathcal{U}$ with formal expansions $U = \sum_{n=0}^{\infty} Z^n U_{[n]}$ and $V = \sum_{n=0}^{\infty} Z^n V_{[n]}$ and let $\Omega = V^*U$. Then the sequence of diagonal operators associated to $\Omega$ is*

$$\Omega_{[m]} = \begin{cases} \sum_{p=0}^{\infty} V_{[p]}^* U_{[p]} & \text{if} \quad m = 0 \\ \sum_{p=m}^{\infty} (V_{[p-m]}^*)^{(m)} U_{[p]} & \text{if} m > 0. \end{cases}$$

*where the convergence is entrywise*

**Proof:** We first assume that $U$ and $V$ are Hilbert–Schmidt operator and write

$$U = \sum_{n=0}^{N} Z^n U_{[n]} + Z^{N+1} R_N \quad \text{and} \quad V = \sum_{n=0}^{N} Z^N V_{[n]} + Z^{N+1} S_N$$

where $R_N, S_N \in \mathcal{U}$. We denote by $P_0(M)$ the main diagonal of an operator $X \in \mathcal{X}$. Then:

$$P_0(\Omega) = \sum_{0}^{N} V_{[n]}^* U_{[n]} + P_0(S_N^* R_N).$$

The operator $S_N^* R_N$ is Hilbert–Schmidt and so is its main diagonal $P_0(S_N^* R_N)$. Furthermore by definition of the Hilbert–Schmidt norm (2.1) and property (2.2) we have

$$\|P_0(S_N^* R_N)\|_{\mathcal{X}_2} \leq \|S_N^* R_N\|_{\mathcal{X}_2} \leq \|S_N^*\| \cdot \|R_N\|_{\mathcal{X}_2} \leq \|S_N\|_{\mathcal{X}_2} \|R_N\|_{\mathcal{X}_2}$$

and so

$$\lim_{N \to \infty} \|P_0(\Omega) - \sum_{0}^{N} V_{[n]}^* U_{[n]}\|_{\mathcal{X}_2} = 0.$$

The same holds also in the operator norm thanks to (2.2).

Now assume that $U \in \mathcal{U}$. Then we first apply the above argument to the operators $UD$ and $VE$ where $D, E \in \mathcal{D}_2$. Then we have for $n = 0$

$$E^* \Omega_{[0]} D = \sum_{p=0}^{\infty} E^* V_{[p]}^* U_{[p]} D$$

where the convergence is in the Hilbert–Schmidt norm. It follows that

$$\Omega_{[0]} = \sum_{p=0}^{\infty} V_{[p]}^* U_{[p]}$$

where the convergence is entrywise. $\square$

**Lemma 4.4** *Let $W \in \mathcal{W}_{NS}$ and let $W = U^* U$ be its Arveson factorization where $U$ and its inverse are upper triangular. Then almost everywhere the limit $U(e^{it}) := \lim_{r \to 1} U(re^{it})$ exists in the strong operator topology, has an upper triangular inverse and satisfies*

$$W(e^{it}) = U(e^{it})^* U(e^{it}). \tag{4.1}$$

**Proof:** First, we note that the operator–valued functions $U(z)$, $U^{-1}(z)$,$U(\overline{z})^*$, are analytic in the open unit disk $\mathbb{D}$ and satisfy

$$\|U(z)\| \leq \|U\|, \ \|U^{-1}(z)\| \leq \|U^{-1}\|, \|U(\overline{z})^*\| \leq \|U\|$$

Therefore, by [20, Theorem A p. 84] the limits $\lim_{r \to 1} U^{\pm 1}(re^{it})$, $\lim_{r \to 1} U(re^{it})^*$ exist almost everywhere in the strong operator topology, and it is easily checked that

$$\begin{aligned} \lim_{r \to 1} U^{-1}(re^{it}) &= U(e^{it})^{-1}, \\ \lim_{r \to 1} U(re^{it})^* &= \left( \lim_{r \to 1} U(re^{it}) \right)^* = U(e^{it})^*. \end{aligned}$$

Note also that if we denote the $n$-th diagonal of $U$ by $U_{[n]}$ then for every $m, n \in \mathbb{Z}$ and $F, G \in \mathcal{D}_2$ it holds almost everywhere that

$$\begin{aligned} \langle U(e^{it}) Z^n F, Z^m G \rangle_{\mathcal{X}_2} &= \lim_{r \to 1} \langle U(re^{it}) Z^n F, Z^m G \rangle_{\mathcal{X}_2} \\ &= \begin{cases} \langle e^{i(m-n)t} Z^{m-n} U_{[m-n]} Z^n F, Z^m G \rangle_{\mathcal{X}_2}, & m \geq n \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

and hence $U(e^{it})$ can be formally written as

$$U(e^{it}) = \sum_{n=0}^{\infty} e^{int} Z^n U_{[n]}.$$

Now, from Proposition 4.3 it follows that the the diagonals of the operators on both sides of 4.1 coincide and hence 4.1 follows. □

**Remark 4.5** *An alternative way to obtain the factorization of* $W(e^{it})$ *is to define* $U(e^{it}) = \Lambda(e^{it}) U \Lambda(e^{it})^{-1}$. *See equation* (2.9).

# 5 Proof of Theorem 1.3

We proceed in a number of steps:

1. By Proposition 3.1 we have a factorization $W(e^{it}) = X(e^{it})^* X(e^{it})$ where the $\mathcal{X}$–valued function $X(z)$ and its inverse are in $\mathcal{W}_+(\mathcal{X})$. At this stage we do not know that in the series
$$X(z) = \sum_{n=0}^{\infty} z^n X_n$$
the Fourier coefficients $X_n$ are of the form $X_n = Z^n X_{[n]}$ for some diagonal operator $X_{[n]}$.

2. In the second step we use Arveson's factorization theorem and Lemma 4.4 to obtain the factorization
$$W(e^{it}) = U(e^{it})^* U(e^{it})$$
where for every real $t$ the operator $U(e^{it})$ and its inverse are upper triangular. The function $U(e^{it})$ is the limit function of the Zadeh transform $U(z)$ of $U$; the function $U(z)$ and its inverse are analytic in $\mathbb{D}$. At this stage we can write
$$U(z) = \sum_{n=0}^{\infty} z^n Z^n U_{[n]}$$
for some diagonal operators (and similarly for $U^{-1}(z)$), but we do not know that $U(z)$ (and its inverse) are in $\mathcal{W}_+(\mathcal{X})$, that is whether the sum
$$\sum_{n=0}^{\infty} \|Z^n U_{[n]}\| = \sum_{n=0}^{\infty} \|U_{[n]}\|$$
converges or not (and similarly for $U^{-1}(z)$).

3. Comparing the two factorizations derived in Steps 1 and 2 we obtain that

$$X(z) = U(z)M$$

where $M$ is a unitary operator. This leads to

$$X_n M^* = Z^n U_{[n]}, \quad n = 0, 1, \cdots$$

and allows to obtain the required factorization result for $W = W(1)$.

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